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A Straightforward Generalization of Diliberto and Straus' Algorithm Does Not Work

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An algorithm for best approximating in the sup-norm a function $f \in C[0,1]^2$ by functions from tensor-product spaces of the form $\pi_k \otimes C[0,1] + C[0,1] \otimes \pi_l$, is considered. For the case $k = l = 0$ the Diliberto and Straus algorithm is known to converge. A straightforward generalization of this algorithm to general k, l is formulated, and an example is constructed demonstrating that this algorithm does not, in general, converge for $k^2 + l^2 > 0$.

The algorithm of Diliberto and Straus for approximating a bivariate function by a sum of univariate ones, proposed in 1951 [1], has been recently investigated in several articles [2–4], where convergence and various properties of the algorithm are studied.

The algorithm, designed for computing the best approximation to $f \in C[0,1]^2$ in the sup-norm from the space

$$M = \{\phi \mid \phi(x, y) \in C[0,1]^2, \phi(x, y) = h(y) + g(x)\}, \tag{1}$$

is of the following form:

$$\begin{aligned} f_0(x, y) &= f(x, y) \\ f_{2n+1}(x, y) &= f_{2n}(x, y) - \tfrac{1}{2}\,[\max_{0 \leqslant \xi \leqslant 1} f_{2n}(\xi, y) \\ &\quad + \min_{0 \leqslant \xi \leqslant 1} f_{2n}(\xi, y)], \qquad n = 0, 1, \ldots, \qquad (2) \\ f_{2n+2}(x, y) &= f_{2n+1}(x, y) - \tfrac{1}{2}\,[\max_{0 \leqslant \eta \leqslant 1} f_{2n+1}(x, \eta) \\ &\quad + \min_{0 \leqslant \eta \leqslant 1} f_{2n+1}(x, \eta)], \qquad n = 0, 1, \ldots. \end{aligned}$$

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It is proved in [1, 3, 4] that $\lim_{n\to\infty} \|f_n\| = \inf_{\phi\in M} \|f-\phi\|$, although the rate of convergence might be extremely slow [2]. Algorithm (2) can be interpreted as a sequence of repeated applications of the operator of one dimensional best approximation by constants to $f(x, y)$, regarded alternately as a function of x and as a function of y. More specifically, let J_x be the operator associating with $f(x, y) \in C[0, 1]^2$ the function $(J_x f)(y) \in C[0, 1]$, with $(J_x f)(y_0)$ the constant of best approximation to $f(x, y_0)$ in the sup-norm on $[0, 1]$, and let J_y be defined similarly with the roles of x, y interchanged. Then (2) can be rewritten as

$$f_0 = f,\ f_{2n+1} = f_{2n} - J_x f_{2n},\ f_{2n+2} = f_{2n+1} - J_y f_{2n+1},\qquad n = 0, 1, 2,.... \tag{3}$$

This formulation suggests a straightforward generalization of algorithm (3), namely, best approximating $f(x, y)$ alternately in the x and y directions by polynomials of degree k and l respectively, in order to obtain a best approximation to $f(x, y)$ from the tensor-product space

$$\begin{aligned} M_{k,l} = \Big\{ &\phi(x, y) | \phi(x, y) \in C[0, 1]^2, \\ &\phi(x, y) = \sum_{j=0}^{k} h_j(y)\, x^j + \sum_{j=0}^{l} g_j(x)\, y^j \Big\} \\ &= \pi_k \otimes C[0, 1] + C[0, 1] \otimes \pi_l. \end{aligned} \tag{4}$$

(π_k denotes the space of all univariate polynomials of degree $\leqslant k$.) With this notation the subspace M in (1) is the tensor-product space $M_{0,0}$. The generalization of algorithm (3) to this more general setting is

$$f_0 = f,\ f_{2n+1} = f_{2n} - J_x^{(k)} f_{2n},\ f_{2n+2} = f_{2n+1} - J_y^{(l)} f_{2n+1},\qquad n = 0, 1, 2,..., \tag{5}$$

where $(J_x^{(k)} f)(x, y_0) = \sum_{j=0}^{k} h_j(y_0)\, x^j$ is the polynomial of best approximation to $f(x,y_0)$ in the sup-norm on $[0, 1]$ from π_k, and where $(J_y^{(l)} f)(x_0, y)$ is similarly defined.

In the following we present a simple example demonstrating that algorithm (5) for general k, l cannot be expected to converge to a best approximation to $f_0(x,y)$.[1] We construct a function $f(x, y)$ such that

$$\|f\| > \inf_{\phi\in M_{0,1}} \|f-\phi\|,$$

[1] Convergence properties of algorithms, which best approximate a function alternately from two subspace, are studied in the case of the L^p-norms, $1 < p < \infty$, by B. Atlestan and F. E. Sullivan in *Rev. Roumaine Math. Pures Appl.* **21** (1976), 125–131. Their result implies the convergence of (5) in $L^p[a, b]^2$, $1 < p < \infty$.

while the functions $\{f_n\}$ generated from it by (5) with $k=0$, $l=1$ satisfy $\|f_n\| = \|f\|$ for all n.

Consider $f(x, y) \in C[0, 1]^2$ subject to the following conditions:

$$
\begin{aligned}
f\left(\frac{i}{4}, \frac{j}{6}\right) &= (-1)^{i+j}, && j = 2i, 2i+1, 2i+2, \qquad i = 0, 1, 2, \\
f\left(\frac{3}{4}, \frac{j}{6}\right) &= (-1)^{j+1}, && j = 0, 5, 6, \\
f\left(1, \frac{2j+1}{6}\right) &= (-1)^{j}, && j = 0, 1, 2, \\
|f(x, y)| &< 1, && \text{elsewhere in } [0, 1]^2.
\end{aligned} \tag{6}
$$

As can be easily observed

$$(J_x^{(0)} f)(x, (i/6)) = 0, \qquad i = 0, 1, \ldots, 6,$$

and

$$(J_y^{(1)} f)((i/4), y) = 0, \qquad i = 0, 1, 2, 3, 4,$$

and both $f - J_x^{(0)} f$ and $f - J_y^{(1)} f$ satisfy (6). Thus algorithm (5) with $k=0$, $l=1$ generates a sequençe $\{f_n\}$ of functions satisfying (6) whenever f_0 satisfies (6), and therefore $\|f_n\| = 1$ for all $n \geqslant 0$.

In order to verify that $\|f\| > \inf_{\phi \in M_{0,1}} \|f - \phi\|$, it is sufficient to show that there does not exist a bounded linear functional $\mu \in (C[0, 1]^2)'$, $\mu \neq 0$, such that

$$\langle \phi, \mu \rangle = 0 \qquad \text{for all} \quad \phi \in M_{0,1}, \tag{7}$$

$$\langle f, \mu \rangle = \|\mu\|. \tag{8}$$

Indeed any $\mu \neq 0$ with property (8) is necessarily of the form

$$
\begin{aligned}
\langle \zeta, \mu \rangle &= \sum_{j=0}^{r} a_j \zeta(x_j, y_j), \qquad \zeta \in C[0, 1]^2, \text{ with } r > 0, \\
a_j f(x_j, y_j) &= |a_j|, \; j = 0, \ldots, r,
\end{aligned} \tag{9}
$$

namely, a linear combination of function values at extremal points of f. Moreover condition (7) implies that μ can be written as a linear combination of first differences in the x direction so as to vanish on all functions of the form $h(y)$, and as a linear combination of second order divided differences in the y direction, so as to vanish on all functions of the form $g_0(x) + g_1(x)\, y$.

These characteristics of μ are consistent with the special structure of the

15 extremal points of f, as given in (6), only if $r = 14$ in (9). Then μ can be written as

$$\langle \zeta, \mu \rangle = \sum_{i=0}^{4} c_i [\ \ ]_i \zeta, \tag{10}$$

where $[\ \ ]_i \zeta$ denotes the second order divided difference of $\zeta((i/4), y)$ at the extremal points of f with $x = i/4$. The sum (10) can be rewritten as a linear combination of first differences in the x direction only if $c_0, \ldots, c_4$ are all equal to zero.

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References

1. S. P. Diliberto and E. G. Straus, On the approximation of a function of several variables by a sum of functions of fewer variables, *Pacific J. Math.* **1** (1951), 195–210.
2. M. von Golitschek and E. W. Cheney, On the algorithm of Diliberto and Straus for approximating bivariate functions by univariate ones, "Numerical Functional Analysis and Optimization," in press.
3. C. T. Kelley, A note on the approximation of functions of several variables by sums of functions of one variable," Report No. 1873, Mathematics Research Center, University of Wisconsin-Madison, August 1978.
4. W. A. Light and E. W. Cheney, On the approximation of a bivariate function by the sum of univariate functions, *J. Approx. Theory* **29** (1980), 305–322.